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On the motive of the group of units of a division algebra

Evgeny Shinder

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Abstract

We consider the algebraic group $GL_1(A)$, where A is a division algebra of prime degree over a field F, and the associated motive in the Voevodsky category of motivic complexes $DM_{-}^{eff}(F)$. We relate the motive of $GL_1(A)$ to the motive of the Čech simplicial scheme $\mathcal{X}$, associated to the Severi-Brauer variety of A, and compute the second differential in the resulting spectral sequence for motivic cohomology.

Keywords: division algebra, Severi-Brauer variety, motivic cohomology

Mathematics Subject Classication 2010: 17A35, 11E57, 14F42, 19E15

1 Introduction

In this paper we consider motives and motivic cohomology of algebraic groups $GL_1(A)$ for a division algebra A of prime degree n over a perfect field F. Motivation to study these groups, as well as more complicated groups $SL_1(A)$ comes from the problems arising in algebraic K-theory, in particular non-triviality of $SK_1(A)$ [S91b], [Me].

It is proved by Biglari [B] that motives of *split* reductive algebraic groups such as $GL_n(F)$ and $SL_n(F)$ are Tate motives. Furthermore, using higher Chern classes in motivic cohomology constructed by Pushin [Pu] one can write down explicit direct sum decompositions for the motives of these two groups with integral coefficients. Proposition 4.2 in the present paper deals with the case of $GL_n(F)$, and the case of $SL_n(F)$ can be treated similarly. Non-split algebraic groups such as $GL_1(A)$ and $SL_1(A)$ are more intricate. We note however that all the complications lie in n-torsion effects ($n = deg(A)$): we are back in the split case if we consider motives with coefficients in $\mathbb{Z}[1/n]$.

The motive of $GL_1(A)$ is closely related to the motive of the Severi-Brauer variety $SB(A)$. We follow an idea of Suslin to break up the motive $M(GL_1(A))$ into two pieces: the first piece is a very simple Tate motive, whereas the second piece is a twisted Tate motive M over $\mathcal{X}$, where $\mathcal{X}$ is the Čech simplicial scheme associated to the Severi-Brauer variety $SB(A)$ (Theorem 4.7). We investigate the structure of the latter motive M using the twisted slice filtration, and compute the second differential in the arising spectral sequence (Theorem 4.9). Using the spectral sequence we compute some lower weight motivic cohomology groups of $GL_1(A)$ (Corollary 4.16) when A is given by a symbol $\theta = (\chi, a)$. We also consider the case of degree 2 algebra where one can write explicit decomposition for $M(GL_1(A))$ (Proposition 4.5).

We now describe the structure of the paper in some detail.

In section 2 we recall the basic facts on central simple algebras, Severi-Brauer varieties and the groups $GL_1(A)$. We formulate and prove Proposition 2.8, which is one of the key geometric tools we use. Some classical references on Severi-Brauer varieties include [A] and [Q].

In section 3 we recall some constructions and results due to Voevodsky [V00], [V03a], [V10a], [V10b], and formulate Propositions 3.5 and 3.6, which constitute the second geometric tool we need and whose proofs are rather straightforward modulo Voevodsky's general machinery. We include a version of the Rost nilpotence theorem (Corollary 3.10), which will not be used in the main body of the text, but fits naturally in the context of motives over $\mathcal{X}$ and the slice filtration and whose proof is this context is also rather straightforward.

In section 4 we consider the motive and motivic cohomology of $GL_1(A)$ by first looking at the split case, then the case of $n = 2$ and finally the general case of prime $n \geq 3$.

Notation: Everywhere in the paper F stands for a perfect field and A is a central simple algebra over F of degree n which is assumed to be prime in Section 4. Throughout the text we keep track of a simple explicit example of a quaternion algebra ($n = 2$) in which case we assume $char(F) \neq 2$. We often use the equality sign to indicate a canonical isomorphism between algebraic varieties or motives.

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2 Varieties associated to central simple algebras

A central simple algebra A of degree n over a field F is an associative unital algebra of dimension n^2 over F that has no nontrivial two-sided ideals and such that the center of A coincides with F.

According to the Wedderburn theorem, A is isomorphic to the matrix algebra $M_n(D)$ over a central division algebra D over F. A is called split if it is isomorphic to $M_n(F)$. It is well known that any central simple algebra splits in some finite separable extension of scalars E/F:

$$A_E = A \otimes_F E \cong M_n(E).$$

Galois descent implies that $det : M_n(F^{sep}) \to F^{sep}$ and $tr : M_n(F^{sep}) \to F^{sep}$ descend to define the so called reduced norm map $Nrd : A \to F$ and the reduced trace map $Trd : A \to F$.

Example 2.1. Let $char(F) \neq 2$. A quaternion algebra $\binom{a,b}{F}$ is defined for $a, b \in F^*$ to be an F-vector space of dimension 4 with the basis $1, i, j, k$ and multiplication $i^2 = a, j^2 = b, ij = -ji = k$. It follows from the Wedderburn theorem, that $\binom{a,b}{F}$ either splits or is a division algebra. Trd and Nrd are the usual trace and norm: $Trd(x + yi + zj + wk) = 2x$, $Nrd(x + yi + zj + wk) = x^2 - ay^2 - bz^2 + abw^2$.

Any central simple algebra of degree two is in fact isomorphic to a quaternion algebra.

Recall that the Severi-Brauer variety $SB(A)$ is a closed subvariety in $Gr(n, A)$ representing the functor which associates to a commutative algebra R over F the set

$$SB(A)(R) = \{right\ ideals\ of\ A \otimes R\ which\ are\ projective\ of\ rank\ n\ over\ R\}.$$

Remark 2.2. Let V be a vector space of dimension n over F, and let A be a split central simple algebra $A = End(V)$. In this case we have a canonical identification

$$SB(End(V)) = \mathbb{P}(V),$$

where a one-dimensional subspace $U \subset V$ corresponds to a right ideal of operators on V whose image is contained in U. In general we have such a description only over a splitting field of A, so that an arbitrary Severi-Brauer variety $SB(A)$ is a twisted form of the projective space $\mathbb{P}^{n-1}$.

Remark 2.3. If $SB(A)$ has a rational point that is to say A has a right ideal I of rank n, then A has to be split. Indeed, the right multiplication action $R_\alpha : I \to I, a \in A$ satisfies $R_{\alpha\beta} = R_\beta R_\alpha$, and the homomorphism

$$R : A \to End(I)^{op} = End(I^*)$$

is an isomorphism by the Schur lemma.

Example 2.4. In the case $A = \binom{a,b}{F}$, $SB(A)$ is isomorphic to a conic in $\mathbb{P}^2$ defined by the equation $x^2 = ay^2 + bz^2$.

By definition, $SB(A)$ being a subvariety in a Grassmannian is endowed with a locally free sheaf $\mathcal{J}$ of rank n with a right A action. $\mathcal{J}$ is a subsheaf of $\mathcal{O}_{SB(A)} \otimes A$. We write $\mathcal{J}^*$ for the dual of $\mathcal{J}$.

Remark 2.5. In the split case $A = End(V)$, $\mathcal{J}$ is identified with $V^* \otimes \mathcal{O}(-1) = \mathcal{H}om(V, \mathcal{O}(-1))$ over $\mathbb{P}(V)$.

Lemma 2.6. *The sheaf of algebras $\mathcal{O}_{SB(A)} \otimes A$ is isomorphic to $\mathcal{E}nd(\mathcal{J}^*)$.*

Proof. The isomorphism is given by the right action of A on $\mathcal{J}$, which is fiberwise given in Remark 2.3. □

We now define the linear algebraic group $GL_1(A)$. For any R is a commutative algebra over F the R-points of this groups are:

$$GL_1(A)(R) = (A \otimes_F R)^* = \{g \in A \otimes_F R : Nrd(g) \neq 0\}$$

One can consider $GL_1(A)$ either an open subscheme in $\mathbb{A}^{n^2}$ or as a form of $GL_n(F)$ twisted by the cocycle defining A.

Example 2.7. For the quaternion algebra $A = \binom{a,b}{F}$, $GL_1(A)$ is an open subscheme in $\mathbb{A}^4$ defined by $x^2 - ay^2 - bz^2 + abw^2 \neq 0$.

Let $E \to T$ be a vector bundle of rank n and consider the associated group scheme $\mathbf{GL}_T(E)$ of local automorphisms of E over T. Let α_E be the tautological automorphism of $p^*(E) = \mathbf{GL}_T(E) \times_T E$ ($p : \mathbf{GL}_T(E) \to T$ is the projection) which maps (g, v) to $(g, g \cdot v)$. Via explicit description of K_1 by Gillet and Grayson [GG], α_E defines an element $[\alpha_E] \in K_1(\mathbf{GL}_T(E))$.

This applies in particular to the case of the trivial rank n bundle $E = F^n$ over a point, in which case we denote the corresponding element in $K_1(GL_n(F))$ by $[\alpha_0]$.

Proposition 2.8. *There is a canonical isomorphism of varieties over $SB(A)$*

$$SB(A) \times GL_1(A) \cong \mathbf{GL}_{SB(A)}(\mathcal{J}^*),$$

where $\mathcal{J}$ is the tautological sheaf of ideals on $SB(A)$.

Furthermore, the tautological class $[\alpha_{\mathcal{J}^*}] \in K_1(\mathbf{GL}(\mathcal{J}^*))$ corresponds under this isomorphism to a class in $K_1(SB(A) \times GL_1(A))$ which in the split case is identified with $[p_1^*(\mathcal{O}(1))] \cdot [p_2^*(\alpha_0)]$ where the product is the standard multiplication for algebraic K-groups $K_0 \otimes K_1 \to K_1$.

Proof. The first assertion follows from Lemma 2.6. Indeed we have a commutative diagram of locally free sheaves

$$\begin{tikzcd} \mathcal{E}nd_{SB(A)}(\mathcal{J}^*) \arrow[r, "det"'] \arrow[d, "\cong"'] & \mathcal{O}_{SB(A)} \arrow[d, equal] \\ \mathcal{O}_{SB(A)} \otimes A \arrow[r, "Nrd"'] & \mathcal{O}_{SB(A)} \end{tikzcd}$$

and we simply need to pass to subvarieties of non-degenerate elements in both rows.

To prove the second assertion, consider the split case $A = End(V)$, and identify $\mathcal{J}^*$ with $V \otimes \mathcal{O}(1)$ by Remark 2.5. Then the isomorphism in question becomes the canonical identification:

$$\mathbb{P}(V) \times GL_1(End(V)) = \mathbf{GL}_{\mathbb{P}(V)}(V \otimes \mathcal{O}) = \mathbf{GL}_{\mathbb{P}(V)}(V \otimes \mathcal{O}(1)),$$

and the claim follows from the following lemma. □

Lemma 2.9. *Let E be a vector bundle and L be a line bundle over the same quasiprojective base T. Then the tautological class $[\alpha_{E\otimes L}] \in K_1(\mathbf{GL}_T(E \otimes L))$ corresponds to $[p^*L] \cdot [\alpha_E] \in K_1(\mathbf{GL}_T(E))$ (p is the projection to T) under the canonical isomorphism of group schemes over T*

$$\mathbf{GL}_T(E) \cong \mathbf{GL}_T(E \otimes L).$$

Proof. Let $\phi : \mathbf{GL}_T(E) \to \mathbf{GL}_T(E \otimes L)$ denote the isomorphism in question. ϕ sends each pair $(t \in T, g \in Aut(E_t))$, to $(t, g \otimes id \in Aut(E_t \otimes L_t))$. Thus it follows that for $\phi^*(\alpha_{E\otimes L}) \in Aut(p^*(E) \otimes p^*(L))$ we have

$$\phi^*(\alpha_{E\otimes L}) = \alpha_E \otimes id_{p^*(L)}. \tag{2.1}$$

Using the Jouanolou trick [J], we may assume that $T = Spec(R)$ is affine, and then E corresponds to a finitely generated projective module M over R. In this setting $\mathbf{GL}_T(E)$ is also affine. Indeed if M is free of rank r, then $\mathbf{GL}_T(E) = T \times GL_r(F)$, and in general M is a direct summand of a trivial R-module, hence $\mathbf{GL}_T(E)$ is closed in some $T \times GL_r(F)$.

In the affine case the claim follows from (2.1) which is the definition of the product $K_0(S) \otimes K_1(S) \to K_1(S)$ (see [Mi], page 27).

□

3 Motivic slice filtration

3.1 Generalities on Voevodsky's categories of motives

We recall some definitions and notation from [V00], [V03a], [V10a]. We work in the category $DM_{-}^{eff}(F)$ of motivic complexes over F as defined in [V00] and in its full subcategory $DM_{\mathcal{X}}$ defined in [V10a] for a simplicial scheme $\mathcal{X}$ over F.

Recall that $DM_{-}^{eff}(F)$ is a tensor triangulated category which admits a covariant monoidal functor from the category of smooth varieties over F

$$M : Sm/F \to DM_{-}^{eff}(F),$$

satisfying the usual properties such as Mayer-Vietoris and localization distinguished triangles.

The category of Tate motives is defined as the full subcategory $DM_{-}^{eff}(F)$ generated by Tate motives $\mathbb{Z}(q)[p], q \geq 0, p \in \mathbb{Z}$. For example $\mathbb{P}^k$ and $\mathbb{A}^k - \{0\}$ have Tate motives:

$$\begin{aligned} M(\mathbb{P}^k) &= \bigoplus_{j=0}^{k} \mathbb{Z}(j)[2j] \\ M(\mathbb{A}^k - \{0\}) &= \mathbb{Z} \oplus \mathbb{Z}(k)[2k-1]. \end{aligned} \tag{3.1}$$

We will frequently use the *Cancellation Theorem* [V10b]

$$Hom_{DM_{-}^{eff}(F)}(M(1), N(1)) = Hom_{DM_{-}^{eff}(F)}(M, N) \tag{3.2}$$

where $M = M \otimes \mathbb{Z}(1)$ and by equality we mean a canonical isomorphism given by the map from the group on the right to the group on the left.

For any smooth variety X the morphism $X \to Spec(F)$ gives rise to a morphism of motives

$$M(X) \to M(Spec(F)) = \mathbb{Z}.$$

One includes this morphism into a distinguished triangle

$$\widetilde{M}(X) \to M(X) \to \mathbb{Z} \to \widetilde{M}(X)[1]. \tag{3.3}$$

A choice of rational point on X (in the case a rational point exists) determines a splitting

$$M(X) = \widetilde{M}(X) \oplus \mathbb{Z}. \tag{3.4}$$

Taking the category $DM_{-}^{eff}(F)$ for granted the motivic cohomology groups and the reduced motivic cohomology groups of degree $p \in \mathbb{Z}$ and weight $q \geq 0$ can be defined to be

$$H^{p,q}(X) := Hom_{DM_{-}^{eff}(F)}(M(X), \mathbb{Z}(q)[p])$$

$$\widetilde{H}^{p,q}(X) := Hom_{DM_{-}^{eff}(F)}(\widetilde{M}(X), \mathbb{Z}(q)[p]),$$

so that distinguished triangles in $DM_{-}^{eff}(F)$ become long exact sequences in motivic cohomology of each weight. It is convenient to define motivic cohomology for $q < 0$ to be identically zero.

If Z is a closed subvariety in X, then we define the motive of X with supports in Z, $M_Z(X)$ as

$$M_Z(X) := C_*(\mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X - Z)).$$

We have a distinguished a triangle of motives

$$M(X\backslash Z) \to M(X) \to M_Z(X) \to M(X\backslash Z)[1].$$

Recall that if Z is smooth of codimension c then we have the Gysin isomorphism ([SV], Theorem 4.10)

$$M_Z(X) \cong M(Z)(c)[2c]. \tag{3.5}$$

Lemma 3.1. *If $T_1 \subset T_0 \subset S$ is a sequence of closed embeddings, then there is a distinguished triangle in $DM_-^{eff}(F)$*

$$M_{T_0\backslash T_1}(S\backslash T_1) \to M_{T_0}(S) \to M_{T_1}(S) \to M_{T_0\backslash T_1}(S\backslash T_1)[1]. \tag{3.6}$$

Proof. The octahedron axiom of triangulated categories ([BBD], Proposition 1.1.11) implies that the commutative square

$$\begin{tikzcd} M(S) \arrow[r, "id"] & M(S) \\ M(S\backslash T_0) \arrow[u] \arrow[r] & M(S\backslash T_1) \arrow[u] \end{tikzcd}$$

can be completed to a 3×3 commutative square with rows and columns being distinguished triangles:

$$\begin{tikzcd} M_{T_0}(S) \arrow[r] & M_{T_1}(S) \arrow[r] & M_{T_0\backslash T_1}(S\backslash T_1)[1] \\ M(S) \arrow[u] \arrow[r] & M(S) \arrow[u] \arrow[r] & 0 \arrow[u] \\ M(S\backslash T_0) \arrow[u] \arrow[r] & M(S\backslash T_1) \arrow[u] \arrow[r] & M_{T_0\backslash T_1}(S\backslash T_1) \arrow[u] \end{tikzcd}$$

Thus $cone(M_{T_0}(S) \to M_{T_1}(S)) \cong M_{T_0\backslash T_1}(S\backslash T_1)[1]$ and we get the distinguished triangle (3.6). □

Recall that the Čech simplicial scheme $\mathcal{X} = \check{C}(SB(A))$ (see [V03a], appendix B) is defined by Voevodsky to consist of $\mathcal{X}_k = SB(A)^{k+1}$ with the face and degeneracy maps taken to be partial projections and diagonals. The canonical morphism $M(\mathcal{X}) \to \mathbb{Z}$ is an isomorphism if $SB(A)$ has an F-point (i.e. if algebra A splits). Recall that $\mathcal{X}$ is an *embedded* simplicial scheme, which means be definition that $M(\mathcal{X}) \otimes M(\mathcal{X}) = M(\mathcal{X})$.

In [V10a], Voevodsky introduces a tensor triangulated category $DM_-^{eff}(\mathcal{X})$ of motives over $\mathcal{X}$ and its close relative $DM_{\mathcal{X}}$, a full subcategory of $DM_-^{eff}(F)$, consisting of objects M satisfying the property that the canonical morphism

$$M \otimes M(\mathcal{X}) \to M \otimes \mathbb{Z} = M$$

is an isomorphism. Note that $M(\mathcal{X})$ is an object in $DM_{\mathcal{X}}$ and we will occasionally write $\mathbb{Z}_{\mathcal{X}}$ for $M(\mathcal{X})$ to emphasize that in the split case $\mathbb{Z}_{\mathcal{X}}$ is canonically isomorphic to $\mathbb{Z}$.

The full embedding $DM_{\mathcal{X}} \subset DM_{-}^{eff}(F)$ admits a right adjoint functor

$$\Phi : DM_{-}^{eff}(F) \to DM_{\mathcal{X}},$$

which on objects is defined to be

$$\Phi(M) = M \otimes M(\mathcal{X})$$

(see Lemma 6.10 in [V10a].)

Remark 3.2. It follows from the adjunction property that for any motive M in $DM_{\mathcal{X}}$, $q \geq 0$, $p \in \mathbb{Z}$

$$H^{p,q}(M, \mathbb{Z}) = Hom_{DM_{-}^{eff}(F)}(M, \mathbb{Z}(q)[p]) \cong Hom_{DM_{\mathcal{X}}}(M, \mathbb{Z}_{\mathcal{X}}(q)[p]).$$

Let $DT(\mathcal{X}) \subset DM_{-}^{eff}(\mathcal{X})$ denote the subcategory of effective Tate motives over $\mathcal{X}$.

3.2 Twisted motivic slice filtration

We need a version of a *slice filtration* on the categories of motivic complexes (see [V10a] and [HK]).

Let M be an object in $DM_{\mathcal{X}}$. For each $q \geq 0$ we define the q-th term of the slice filtration of M to be:

$$\nu_{\mathcal{X}}^{\geq q} M = \underline{Hom}_{DM_{-}^{eff}(F))}(\mathbb{Z}(q), M)(q) \otimes \mathbb{Z}_{\mathcal{X}}.$$

The internal Hom-object above exists by [V00], Proposition 3.2.8.

Remark 3.3. It is easy to see using the adjunction property that

$$\underline{Hom}_{DM_{-}^{eff}(F)}(\mathbb{Z}(q), M)(q) \otimes M(\mathcal{X})$$

is in fact isomorphic to

$$\underline{Hom}_{DM_{\mathcal{X}}}(\mathbb{Z}_{\mathcal{X}}(q), M)(q).$$

It is also easy to see that for Tate motives our slice filtration coincides with the one from [V10a].

We define $\nu_{\mathcal{X}}^{q}$ as the cone in the distinguished triangle

$$\nu_{\mathcal{X}}^{\geq q+1}(M) \to \nu_{\mathcal{X}}^{\geq q}(M) \to \nu_{\mathcal{X}}^{q}(M) \to \nu_{\mathcal{X}}^{\geq q+1}(M)[1].$$

The triangulated functors $\{\nu_{\mathcal{X}}^{\geq q}\}$ commute with extension of scalars and for each $k, j \geq 0$ satisfy

$$\nu_{\mathcal{X}}^{\geq k+j}(M(j)) = \nu_{\mathcal{X}}^{\geq k}(M)(j).$$

Remark 3.4. For a split Tate motive $M = \oplus_{p,q} \mathbb{Z}_{\mathcal{X}}(p)[q]^{\oplus a_{p,q}}$ we have

$$\nu_{\mathcal{X}}^{\geq k}(M) = \oplus_{p \geq k, q} \mathbb{Z}_{\mathcal{X}}(p)[q]^{\oplus a_{p,q}}$$

and

$$\nu_{\mathcal{X}}^{k}(M) = \oplus_{q} \mathbb{Z}_{\mathcal{X}}(k)[q]^{\oplus a_{k,q}}.$$

The following two propositions provide geometric criteria for motives to lie in $DM_{\mathcal{X}}$ and $DT(\mathcal{X})$ respectively.

Proposition 3.5. *Let T be a variety over F.*

1. If T is smooth and for each generic point η of T $A_{F(\eta)}$ is a split algebra then $M(T)$ lies in $DM_{\mathcal{X}}$.

2. Let $T \subset S$ be a closed embedding of T into a smooth variety S. If for each scheme-theoretic point $z \in T$ $A_{F(z)}$ is a split algebra then $M_T(S)$ lies in $DM_{\mathcal{X}}$.

Proof. (1) We need to show that $M(T) \otimes C = 0$ where $C = cone(M(\mathcal{X}) \to \mathbb{Z})$. This follows from [V03a], Lemma 4.5.

(2) We filter T by closed subvarieties

$$T_N \subset T_{N-1} \subset \cdots \subset T_1 \subset T_0 = T \subset S$$

where $T_k \backslash T_{k+1}$ are nonsingular. We prove by the descending induction on k that $M_{T_k}(S)$ is an object in $DM_{\mathcal{X}}$. The base case $k = N$ follows from (1) and the Gysin isomorphism (3.5): since T_N is smooth,

$$M_{T_N}(S) \cong M(T_N)(c)[2c] \in DM_{\mathcal{X}}.$$

For the induction step, we use the distinguished triangle of Lemma 3.1:

$$M_{T_k \backslash T_{k+1}}(S \backslash T_{k+1}) \to M_{T_k}(S) \to M_{T_{k+1}}(S) \to M_{T_k \backslash T_{k+1}}(S \backslash T_{k+1})[1]$$

Since by induction hypothesis and by applying the first claim of the Lemma again, $M_{T_{k+1}}(S)$ and $M_{T_k \backslash T_{k+1}}(S \backslash T_{k+1})$ lie in $DM_{\mathcal{X}}$, $M_{T_k}(S)$ also lies in $DM_{\mathcal{X}}$.

□

Proposition 3.6. *Let M be an object in $DM_{\mathcal{X}}$. Assume that $M_{F(SB(A))}$ is a split Tate motive of the form $\oplus_{p,q} \mathbb{Z}(p)[q]^{\oplus a_{p,q}}$. Then the slice filtration of M in $DM_{\mathcal{X}}$ has successive cones which are split Tate motives*

$$\nu^p_{\mathcal{X}}(M) = \oplus_q \mathbb{Z}_{\mathcal{X}}(p)[q]^{\oplus a_{p,q}}.$$

In particular, M is a mixed Tate motive over $\mathcal{X}$.

For the proof we need the following lemma, which we borrow from [S].

Lemma 3.7. *For any M from $DM^{eff}_{-}(F)$ and $p \in \mathbb{Z}$ the extension of scalars $H^{p,0}(M) \to H^{p,0}(M_{F(SB(A))})$ is an isomorphism.*

Proof. It is sufficient to prove the statement in the case $M = M(S)[j]$ where S is a smooth connected scheme over F. In this case the homomorphism in question takes the form:

$$H^{p-j,0}(S) \to H^{p-j,0}(S_{F(SB(A))}),$$

and both groups are equal 0 for $p \neq j$.

S is connected, and $SB(A)$ being geometrically irreducible has separably generated function field $F(SB(A))$, hence $S_{F(SB(A))}$ is connected as well. Therefore if $p = j$ both cohomology groups in question are isomorphic to $\mathbb{Z}$ with the map being the identity.

□

Proof of Proposition 3.6. Let $\nu^p_{\mathcal{X}} M = c_p(M)(p)$. Then

$$\begin{aligned} &Hom(\nu^p_{\mathcal{X}} M, \mathbb{Z}_{\mathcal{X}}(p)[q]) \\ &= Hom(c_p(M), \mathbb{Z}_{\mathcal{X}}[q]) \text{ by the Cancellation Theorem (3.2)} \\ &= H^{q,0}(c_p(M), \mathbb{Z}) \text{ by Remark 3.2} \\ &= H^{q,0}(c_p(M_{F(SB(A))}), \mathbb{Z}) \text{ by Lemma 3.7} \\ &= H^{q,0}(\oplus_r \mathbb{Z}[r]^{\oplus a_{p,r}}, \mathbb{Z}) \text{ by Remark 3.4} \\ &= \mathbb{Z}^{\oplus a_{p,q}}. \end{aligned}$$

Therefore there exists a morphism $\phi_p : \nu^p_{\mathcal{X}} M \to \oplus_q \mathbb{Z}_{\mathcal{X}}(p)[q]^{\oplus a_{p,q}}$ such that ϕ_p becomes an isomorphism after scalar extension to $F(SB(A))$. This implies that $cone(\phi_p)_{F(SB(A))} = 0$, so that $cone(\phi_p) = cone(\phi_p) \otimes M(\mathcal{X}) = 0$ by [V03a], Lemma 4.5, and thus ϕ_p is an isomorphism. □

Remark 3.8. As the example of $M = M(SB(A))$ shows, M itself is not always a *split* Tate motive. Indeed it is a result of Karpenko [K] that for a division algebra A, $M(SB(A))$ is indecomposable[1].

Example 3.9. Let $A = \binom{a,b}{F}$, and let $M_{a,b} = M(SB(A))$ be the Rost motive. In this case the slice filtration is the distinguished triangle

$$\mathbb{Z}_{\mathcal{X}}(1)[2] \to M_{a,b} \to \mathbb{Z}_{\mathcal{X}} \to \mathbb{Z}_{\mathcal{X}}(1)[3]$$

from [V03a], Theorem 4.4.

As a corollary of Proposition 3.7 and the existence of the slice filtration we easily deduce the following version of the Rost nilpotence theorem (cf [CGM], Cor. 8.4 and [R], Cor. 10).

Proposition 3.10. *Let M be a Tate motive of the form $M = \bigoplus_{k=0}^n \mathbb{Z}(i_k)[2\, i_k]$. Let*

$$f : M(SB(A)) \otimes M \to M(SB(A)) \otimes M$$

be a morphism of motives. If $f_{F(SB(A))}$ is an isomorphism then f is an isomorphism.

Proof. Consider the slice filtration on $M(SB(A)) \otimes M$. By Lemma 3.6 the slices $\nu^p_{\mathcal{X}}(M(SB(A)) \otimes M)$ are equal to $\mathbb{Z}_{\mathcal{X}}(p)[2p]^{\oplus a_p}$, for some $a_p \geq 0$. The morphisms induced on the slices are given by matrices with coefficients in $Hom(\mathbb{Z}_{\mathcal{X}}, \mathbb{Z}_{\mathcal{X}})$, and this group is identified with $\mathbb{Z}$ using Remark 3.2 and Lemma 3.7.

□

The slice filtration gives rise to an exact couple for each weight j

$$E^{p,q} = H^{p+q}(\nu^q_{\mathcal{X}}(M), \mathbb{Z}(j)),$$

$$D^{p,q} = H^{p+q}(M, \nu^{\geq q+1}_{\mathcal{X}}(M), \mathbb{Z}(j)),$$

[1]This result is proved in [K] in the category of Chow motives $CHM(F)$, which is a full subcategory of $DM^{eff}_{-}(F)$ (see [V00], Proposition 2.1.4 and Remark after Corollary 2.1.5 for the statement in characteristic zero; for an arbitrary perfect field one also needs [V03b]). $CHM(F)$ is Karoubian, therefore any direct sum decomposition of $M(SB(A))$ in $DM^{eff}_{-}(F)$ would lead to a decomposition in $CHM(F)$.

$$\cdots \to D^{p+1,q-1} \to D^{p,q} \to E^{p,q} \to D^{p+2,q-1} \to \dots$$

and the corresponding spectral sequence

$$E_2^{p,q} = H^{p+q}(\nu_{\mathcal{X}}^q(M), \mathbb{Z}(j)) \Rightarrow H^{p+q}(M, \mathbb{Z}(j)), \tag{3.7}$$

with the differential $d_2 : H^{p+q-1}(\nu_{\mathcal{X}}^{q+1}M, \mathbb{Z}(j)) \to H^{p+q}(\nu_{\mathcal{X}}^q M, \mathbb{Z}(j))$ induced by the q-th connecting morphism $\partial_{q,M}$ given by the composition of morphisms forming the slice filtration:

$$\partial_{q,M} : \nu_{\mathcal{X}}^q(M) \to \nu_{\mathcal{X}}^{\geq q+1}(M)[1] \to \nu_{\mathcal{X}}^{q+1}(M)[1]. \tag{3.8}$$

4 The motive of $GL_1(A)$

4.1 The split case

We consider the group variety $GL_n(F)$ over a field F. To give an explicit description of $M(GL_n(F))$ we use the higher Chern classes $c_{j,i}$ for motivic cohomology

$$c_{j,i} : K_j(X) \to H^{2i-j,i}(X),\ i, j \geq 0. \tag{4.1}$$

Note that the ordinary Chern classes are $c_i = c_{0,i}$. In the computations in this section we use $c_{1,i}$.

We recall the construction of the higher Chern classes using $\mathbb{A}^1$-motivic homotopy category $\mathcal{H}_\bullet(F)$ of Morel and Voevodsky. The construction we give is essentially the same as in [Pu] but we follow the approach of [Ri]. The basic references for $\mathbb{A}^1$-homotopy is [MV], see [V98] for a short introduction.

In the homotopy category of pointed spaces $\mathcal{H}_\bullet(F)$ both higher algebraic K-theory and motivic cohomology are representable: if X is a smooth variety over F, then

$$K_j(X) = Hom_{\mathcal{H}_\bullet(F)}(\Sigma^j X_+, \mathbb{Z} \times \mathbf{Gr})$$
$$H^{2i-j,i}(X) = Hom_{\mathcal{H}_\bullet(F)}(\Sigma^j X_+, \mathbf{H}(\mathbb{Z}(i), 2i))$$

in analogy with the situation in topology. If in addition we define

$$\widetilde{K}_j(X) = Hom_{\mathcal{H}_\bullet(F)}(\Sigma^j X_+, \mathbf{Gr})$$

then

$$K_0(X) = \widetilde{K}_0(X) \oplus \mathbb{Z}$$
$$K_j(X) = \widetilde{K}_j(X),\ j > 0.$$

The Chern classes (4.1) are induced by a morphism of pointed spaces

$$\mathbf{c}_i : \mathbb{Z} \times \mathbf{Gr} \to \mathbf{H}(\mathbb{Z}(i), 2i)$$

(cf [Ri], Theorem 6.2.1.2). It follows from this definition that $c_{j,i}$ are natrual transformations of functors. We will need the following product formula.

Proposition 4.1. *Let X be a smooth variety. If $\lambda \in Pic(X) = H^{2,1}(X)$ and $\alpha \in K_j(X)$, $j > 0$ or $\alpha \in \widetilde{K}_0(X)$, then*

$$\begin{aligned} c_{j,i}(\lambda \cdot \alpha) &= \sum_{l=0}^{k-1} (-1)^l \binom{i-1}{l} \lambda^l \, c_{j,i-l}(\alpha) \\ &= c_{j,i}(\alpha) - (i-1)\lambda \, c_{j,i-1}(\alpha) + \frac{(i-1)(i-2)}{2} \lambda^2 \, c_{j,i-2}(\alpha) + \cdots + (-1)^{i-1} \lambda^{i-1} \, c_{j,1}(\alpha) \end{aligned} \tag{4.2}$$

(the formula is independent of j).

Proof. Assume that α is an element in $K_0(X)$ of virtual rank r. Using the splitting principle it is easy to see that

$$c_i(\lambda \cdot \alpha) = \sum_{l=0}^{k} \binom{r-i+l}{i} \lambda^l c_{i-l}(\alpha) \in H^{2i,i}(X, \mathbb{Z}). \tag{4.3}$$

In particular, if $\alpha \in \widetilde{K}_0(X)$, so that $r = 0$, then $\binom{-i+l}{l} = (-1)^l \binom{i-1}{l}$ and

$$c_i(\lambda \cdot \alpha) = \sum_{l=0}^{k-1} (-1)^l \binom{i-1}{l} \lambda^l c_{i-l}(\alpha). \tag{4.4}$$

To extend the formula (4.4) to K_j we use the method of [Ri]: we consider two natural transformations of presheaves on the category Sm/X of smooth schemes over X:

$$\theta_j, \theta_j' : K_j(-) \to H^{2i-j,i}(-)$$

given for $p : Y \to X$ by

$$\alpha \in K_j(Y) \mapsto c_{j,i}(p^*(\lambda) \cdot \alpha)$$

and

$$\alpha \in K_j(Y) \mapsto \sum_{l=0}^{k-1} (-1)^l \binom{r(\alpha) - i + l}{l} p^*(\lambda)^l \, c_{j,i-l}(\alpha)$$

respectively. Note that the virtual rank $r(\alpha)$ can be non-zero only for $\alpha \in K_0(X)$.

By construction θ_j, θ_j' are induced by two morphisms

$$\Theta, \Theta' : \mathbb{Z} \times \mathbf{Gr} \to \mathbf{H}(\mathbb{Z}(i), 2i)$$

(independent of $j \geq 0$). By [Ri] Theorem 1.1.6 to check that $\Theta = \Theta'$ is suffices to show that $\theta_0 = \theta_0' : K_0(-) \to H^{2i,i}(-)$. This holds by (4.3).

□

From now on in this section we only work with Chern classes

$$c_i := c_{1,i} : K_1(-) \to H^{2i-1,i}(-).$$

If $\alpha \in K_1(X)$ and I is a multi-index

$$I = \{1 \leq i_1 < \cdots < i_r \leq n\}$$

we let

$$|I| = i_1 + \cdots + i_r$$
$$l(I) = r$$

and

$$c_I(\alpha) = c_{i_1}(\alpha) \cdot \cdots \cdot c_{i_r}(\alpha) \in H^{2|I|-l(I),|I|}(X).$$

Proposition 4.2. *The motive $M(GL_n(F))$ admits the following direct sum decomposition:*

$$M(GL_n(F)) \cong \bigoplus_I \mathbb{Z}(|I|)[2|I| - l(I)],$$

where the morphism

$$M(GL_n(F)) \to \mathbb{Z}(|I|)[2|I| - l(I)]$$

corresponds to the class

$$c_I(\alpha) \in H^{2|I|-l(I),|I|}(GL_n(F)),$$

$[\alpha]$ is the tautological class in $K_1(GL_n(F))$ defined in the paragraph preceding Proposition 2.8.

Proof. We define the morphism

$$\phi : M(GL_n(F)) \to \bigoplus_I \mathbb{Z}(|I|)[2|I| - l(I)]$$

using the classes c_I. We claim that ϕ is an isomorphism.

First note, that for any reductive split group G over F the motive $M(G)$ is a Tate motive [B], Proposition 4.2. Therefore by the Yoneda lemma it is sufficient to check that ϕ induces isomorphism on the motivic cohomology groups.

According to [Pu], Lemma 13, motivic cohomology of $GL_n(F)$ is generated freely by the classes $c_I(\alpha)$ and the statement follows.

□

We also need a relative version of Proposition 4.2.

Proposition 4.3. *Let $E \to T$ be a vector bundle of rank n, and let α_E be the tautological class in $K_1(\mathbf{GL}(E))$. The motive $M(\mathbf{GL}(E))$ admits the following decomposition:*

$$M(\mathbf{GL}(E)) = \bigoplus_I M(T)(|I|)[2|I| - l(I)]$$

where the morphism

$$M(\mathbf{GL}(E)) \to M(T)(|I|)[2|I| - l(I)]$$

is the composition

$$M(\mathbf{GL}(E)) \to M(\mathbf{GL}(E)) \otimes M(\mathbf{GL}(E)) \to M(\mathbf{GL}(E))(|I|)[2|I| - l(I)] \to$$
$$M(T)(|I|)[2|I| - l(I)]$$

of multiplication by the class

$$c_I(\alpha_E) \in H^{2|I|-l(I),|I|}(\mathbf{GL}(E)).$$

followed by the canonical projection.

Proof. The statement follows from Proposition 4.2 and the Mayer-Vietoris distinguished triangle.

□

4.2 The case $n = 2$

Let $A = \binom{a,b}{F}$, and let $C = SB(A)$ be the norm conic. In this case $GL_1(A)$ is the complement to $Q \subset \mathbb{A}^4 - \{0\}$ in $\mathbb{A}^4 - \{0\}$, where

$$Q = \{(x, y, z, w) \in \mathbb{A}^4 - \{0\} : x^2 - ay^2 - bz^2 + abw^2 = 0\}.$$

Lemma 4.4. $M(Q) = M(C) \oplus M(C)(2)[3]$.

Proof. First note that the projective quadric $\{x^2 - ay^2 - bz^2 + abw^2 = 0\} \subset \mathbb{P}^3$ is isomorphic to $C \times C$. Indeed we have the Segre embedding

$$C \times C = SB(A) \times SB(A) \cong SB(A) \times SB(A^\vee) \to SB(A \otimes A^\vee) \cong SB(End_F(A)) \cong \mathbb{P}(A) \cong \mathbb{P}^3$$

and the image consists of elements of rank 1 and thus the image is given by one homogeneous equation $Nrd(\alpha) = x^2 - ay^2 - bz^2 + abw^2 = 0$.

It can be proved analogously to Proposition 2.8 that $C \times C$ is a projective line bundle over C, therefore

$$M(C \times C) = M(C) \oplus M(C)(1)[2].$$

Q over $C \times C$ is the complement to the zero section in the line bundle $\mathcal{O}(-1)$. We have a distinguished triangle

$$M(C)(1)[1] \oplus M(C)(2)[3] \to M(Q) \to M(C) \oplus M(C)(1)[2] \to M(C)(1)[2] \oplus M(C)(2)[4],$$

with the third morphism being the natural one and the claim follows since after separating the summand $M(C)(1)[2]$ the resulting distinguished triangle is split.

□

Proposition 4.5. *There is a decomposition*

$$M(GL_1(A)) = \mathbb{Z} \oplus M(C)(1)[1] \oplus \mathbb{Z}_{a,b}(3)[4],$$

where we temporarily use the notation $\mathbb{Z}_{a,b}$ for the cone of the canonical morphism $\mathbb{Z}(1)[2] \to M(C)$ corresponding to the fundamental class $[C] \in CH^0(C) = CH_1(C)$.

Proof. Consider the distinguished triangle corresponding to the open embedding

$$GL_1(A) \subset \mathbb{A}^4 - \{0\} :$$

$$M_Q(\mathbb{A}^4 - \{0\})[-1] \to \widetilde{M}(GL_1(A)) \to \widetilde{M}(\mathbb{A}^4 - \{0\}) \to M_Q(\mathbb{A}^4 - \{0\}). \tag{4.5}$$

We have $\widetilde{M}(\mathbb{A}^4 - \{0\}) = \mathbb{Z}(4)[7]$ and also

$$M_Q(\mathbb{A}^4 - \{0\}) = M(Q)(1)[2] = M(C)(1)[2] \oplus M(C)(3)[5],$$

with the first equality being Gysin isomorphism and the second one comes from Lemma 4.4.

The distinguished triangle (4.5) now can be rewritten as:

$$M(C)(1)[1] \oplus M(C)(3)[4] \to \widetilde{M}(GL_1(A)) \to \mathbb{Z}(4)[7] \to M(C)(1)[2] \oplus M(C)(3)[5].$$

By dimension reasons $Hom(\mathbb{Z}(4)[7], M(C)(1)[2]) = 0$, therefore

$$\widetilde{M}(GL_1(A)) = M(C)(1)[1] \oplus cone(\mathbb{Z}(4)[7] \to M(C)(3)[5])[-1].$$

The morphism $\mathbb{Z}(4)[7] \to M(C)(3)[5]$ corresponds to a class in $CH_1(C) = CH^0(C)$ which can be computed after passing to a splitting field by Lemma 3.7. In the split case we can verify that the morphism in question corresponds via the Cancellation Theorem (3.2) to the fundamental class $[C]$.

□

Remark 4.6. Note that in the split case $C = \mathbb{P}^1$ and $\mathbb{Z}_{a,b} = \mathbb{Z}$ so that the we have

$$M(GL_2(F)) = \mathbb{Z} \oplus \mathbb{Z}(1)[1] \oplus \mathbb{Z}(2)[3] \oplus \mathbb{Z}(3)[4]$$

in agreement with Proposition 4.2.

4.3 The general case

We assume $n \geq 3$ is a prime. Let Z be the complement of $GL_1(A)$ in $\mathbb{A}^{n^2} - \{0\}$, i.e. the subvariety in $\mathbb{A}^{n^2} - \{0\}$ given by equation $Nrd_A = 0$. Let $M = M_Z(\mathbb{A}^{n^2} - \{0\})[-1]$ be a motive with supports which is determined by the distinguished triangle

$$M \to M(GL_1(A)) \to M(\mathbb{A}^{n^2} - \{0\}) \to M[1]. \tag{4.6}$$

We concentrate on studying the motive M.

Theorem 4.7. *1. For $j < n^2$ and $p \in \mathbb{Z}$ we have a canonical isomorphism*

$$\widetilde{H}^{p,j}(GL_1(A)) \to H^{p,j}(M).$$

2. If A splits, then we have a decomposition

$$M = \widetilde{M}(GL_1(A)) \oplus \mathbb{Z}(n^2)[2n^2 - 2] = \bigoplus_{I \neq \emptyset} \mathbb{Z}_{\mathcal{X}}(|I|)[2|I| - l(I)] \oplus \mathbb{Z}(n^2)[2n^2 - 2].$$

3. M is an object in $DT(\mathcal{X})$ and the slices of the slice filtration are given by:

$$\nu^q_{\mathcal{X}}(M) = \begin{cases} \bigoplus_{|I|=q} \mathbb{Z}_{\mathcal{X}}(q)[2q - l(I)], & 1 \leq q \leq \frac{n(n+1)}{2} \\ \mathbb{Z}_{\mathcal{X}}(n^2)[2n^2 - 2], & q = n^2 \\ 0, & otherwise \end{cases}$$

Proof. Motivic cohomology of $GL_1(A)$ and that of M are related via the long exact sequence

$$\widetilde{H}^{p,j}(\mathbb{A}^{n^2} - \{0\}) \to \widetilde{H}^{p,j}(GL_1(A)) \to H^{p,j}(M) \to \widetilde{H}^{p+1,j}(\mathbb{A}^{n^2} - \{0\}),$$

and the first claim follows since using (3.1) we see that

$$\widetilde{H}^{p,j}(\mathbb{A}^{n^2} - \{0\}) = H^{p,j}(\mathbb{Z}(n^2)[2n^2 - 1]) = 0$$

for $j < n^2$ and any $p \in \mathbb{Z}$.

If the algebra A is split, then in the distinguished triangle

$$M \to \widetilde{M}(GL_n(F)) \to \widetilde{M}(\mathbb{A}^{n^2} - \{0\}) \to M[1]$$

the second morphism is zero, since as a simple computation using Proposition 4.2 shows, $Hom(\widetilde{M}(GL_n(F)), \widetilde{M}(\mathbb{A}^{n^2} - \{0\})) = 0$. The triangle splits yielding the first equality in the second claim. The second equality follows from Proposition 4.2.

To prove the third claim note that any point of $z \in Z$ splits A: $A_{F(z)}$ has a non-zero non-invertible element (given by z) therefore $A_{F(z)}$ is not a division algebra, and since we assume that the degree n of A is prime, $A_{F(z)}$ splits. The third claim now follows from Propositions 3.5 and 3.6.

□

We investigate the slice spectral sequence (3.7) for the motive M. If we consider the weights $j < n^2$, then by Theorem 4.7 the spectral sequence in question actually converges to $\widetilde{H}^{*,j}(GL_1(A))$. It also follows from Theorem 4.7 that the second page E_2 of the spectral sequence will be formed from the motivic cohomology groups of $\mathbb{Z}_{\mathcal{X}}$. The second differential will be naturally given in terms of cohomology classes in $H^{3,1}(\mathbb{Z}_{\mathcal{X}})$.

Lemma 4.8. *If A is non-split, then there is a canonical isomorphism*

$$H^{3,1}(\mathbb{Z}_{\mathcal{X}}) = \mathbb{Z}/n,$$

and if A is split, $H^{3,1}(\mathbb{Z}_{\mathcal{X}}) = 0$.

Proof. Assume first that A is non-split. The isomorphism

$$H^{3,1}(\mathbb{Z}_{\mathcal{X}}) \cong Ker(res : H^2_{et}(F, \mu_n) \to H^2_{et}(F(SB(A)), \mu_n)).$$

is established in [MS], Proposition 1.4 (the assumption made in [MS] that the class $[A] \in {}_nBr(F)$ is a symbol does not play a role in the proof).

On the other hand for any field $H^2_{et}(F, \mu_n)$ is canonically isomorphic to the n-torsion of the Brauer group $Br(F)$, and the kernel of the restriction map $Br(F) \to Br(F(X))$ is generated by the class of algebra A by the classical theorem of Amitsur. Since the period of A is equal to n the statement of the Lemma follows.

If A is split, then $\mathbb{Z}_{\mathcal{X}} = M(Spec(F))$ and we have $H^{3,1}(\mathbb{Z}_{\mathcal{X}}) = H^{3,1}(Spec(F)) = 0$ by standard vanishing theorems for motivic cohomology. □

We denote the generator of $H^{3,1}(\mathbb{Z}_{\mathcal{X}}) = \mathbb{Z}/n$ corresponding to $[A] \in Br(F)$ in the proof of Lemma 4.8 by δ. This notation is consistent with [MS], 1.5.

Let $1 \leq q < \frac{n(n+1)}{2}$. The second differential d_2 in the slice spectral sequence for M is induced by the morphism of motives (3.8)

$$\begin{tikzcd}
\bigoplus_{|I|=q} \mathbb{Z}_{\mathcal{X}}(q)[2q - l(I)] \arrow[r, equal] & \nu^q_{\mathcal{X}}(M) \arrow[d, "\partial_q"] & \\
& \nu^{q+1}_{\mathcal{X}}(M)[1] \arrow[r, equal] & \bigoplus_{|J|=q+1} \mathbb{Z}_{\mathcal{X}}(q+1)[2q+3-l(J)]
\end{tikzcd}$$

with components

$$\partial_{I,J} : \mathbb{Z}_{\mathcal{X}}(q)[2q - l(I)] \to \mathbb{Z}_{\mathcal{X}}(q+1)[2q+3-l(J)] \tag{4.7}$$

corresponding to multi-indices $I, |I| = q$ and $J, |J| = q+1$. Each morphism $\partial_{I,J}$ determines a class

$$\partial_{I,J} \in H^{3-l(J)+l(I),1}(\mathbb{Z}_{\mathcal{X}}).$$

Theorem 4.9. *Let A be a division algebra of prime degree $n \geq 3$.*

1. The morphism $\partial_{I,J}$ in (4.7) is zero unless $l(I) = l(J)$ and the sequence J is obtained from the sequence I by increasing exactly one index by one.

2. If A is a division algebra, then there exists $c = c(A) \in \mathbb{Z}/n$, $c \neq 0$ with the following property: if the sequence J is obtained from the sequence I by increasing an index i_t by one, then

$$\partial_{I,J} = i_t \cdot c \cdot \delta \in H^{3,1}(\mathbb{Z}_{\mathcal{X}}).$$

Finally, if A is a split algebra, then all $\partial_{I,J} = 0$.

Proof. The idea of the proof is to compare the slice filtration of M with that of the motive of the Severi-Brauer variety $M(SB(A))$. More precisely we will express all potentially non-vanishing $\partial_{I,J}$ in terms of the 0-th connecting morphism $\partial' := \partial_{0,M(SB(A))}$ (3.8) in the slice filtration of $M(SB(A))$.

We fix a weight q and a multi-index

$$I = \{i_1, \ldots, i_r\}$$

such that

$$|I| = \sum_{t=1}^{r} i_t = q.$$

Consider the motive $M(SB(A) \times GL_1(A))$. According to Proposition 2.8

$$SB(A) \times GL_1(A) = \mathbf{GL}_{SB(A)}(\mathcal{J}^*),$$

and Proposition 4.3 implies that $M(SB(A) \times GL_1(A))$ admits a direct summand

$$M(SB(A))(q)[2q - r] \subset M(SB(A) \times GL_1(A))$$

corresponding to the class $c_I(\alpha_E)$. We denote this embedding by ι and consider the diagram

$$\begin{tikzcd} M(SB(A))(q)[2q-r] \arrow[r, "\iota"] \arrow[drr, "\psi"'] & M(\mathbf{GL}_{SB(A)}(\mathcal{J}^*)) \arrow[r, equal] & M(GL_1(A) \times SB(A)) \arrow[d] \\ & & M(GL_1(A)) \end{tikzcd} \tag{4.8}$$

Lemma 4.10. *There exists a unique morphism ϕ which fits in the diagram:*

$$\begin{tikzcd} M(SB(A))(q)[2q-r] \arrow[d, dotted, "\phi"'] \arrow[dr, "\psi"] & \\ M \arrow[r] & M(GL_1(A)) \end{tikzcd}$$

Proof. From the distinguished triangle (4.6) defining M we see that it is sufficient to show that

$$Hom(M(SB(A))(q)[2q-r], M(\mathbb{A}^{n^2}-\{0\})[\epsilon]) = 0,$$

for $\epsilon = 0, -1$. We have $M(\mathbb{A}^{n^2}-\{0\})[\epsilon] = \mathbb{Z}[\epsilon] \oplus \mathbb{Z}(n^2)[2n^2-1+\epsilon]$ so that

$$Hom(M(SB(A))(q)[2q-r], M(\mathbb{A}^{n^2}-\{0\})[\epsilon]) = H^{\epsilon-(2q-r),-q}(SB(A)) \oplus H^{2n^2-1+\epsilon-(2q-r),n^2-q}(SB(A)).$$

Now both cohomology groups are zero: the first one because it is of strictly negative weight, and second one because the degree is greater than weight plus dimension:

$$2n^2-1+\epsilon-(2q-r)-(n^2-q) = n^2-q+r-1+\epsilon > dim(SB(A)) = n-1,$$

under the assumptions $n \geq 3$ and $q < \frac{n(n+1)}{2}$. $\square$

The morphism

$$\phi : M(SB(A))(q)[2q-r] \to M$$

that we have just defined induces a morphism on the slice filtrations of the source and target motives. For each $q \leq k \leq q+n-1$ we get a commutative diagram

$$\begin{tikzcd}
\nu^k_{\mathcal{X}}(M(SB(A))(q)[2q-r]) \arrow[r, "\nu^k_{\mathcal{X}}(\phi)"'] \arrow[d, equal] & \nu^k_{\mathcal{X}}(M) \arrow[d, equal] \\
\mathbb{Z}_{\mathcal{X}}(k)[2k-r] \arrow[r, "\oplus\nu^k_{\mathcal{X}}(\phi)_J"'] & \bigoplus_{|J|=k} \mathbb{Z}_{\mathcal{X}}(k)[2k-l(J)]
\end{tikzcd}$$

where the equality on the left follows from Proposition 3.6 and the equality on the right is established by Theorem 4.7.

Each $\nu^k_{\mathcal{X}}(\phi)_J$, $|J| = k$ is an element in the group

$$Hom(\mathbb{Z}_{\mathcal{X}}(k)[2k-r]\,\mathbb{Z}_{\mathcal{X}}(k)[2k-l(J)]) = Hom(\mathbb{Z}_{\mathcal{X}}, \mathbb{Z}_{\mathcal{X}}[r-l(J)]) = H^{r-l(J),0}(\mathbb{Z}_{\mathcal{X}})$$

(the second isomorphism comes from Remark 3.2). By Lemma 3.7 the latter cohomology group is isomorphic to $\mathbb{Z}$ when $l(J) = r$ and is zero otherwise. Thus in what follows each symbol $\nu^k_{\mathcal{X}}(\phi)_J$ will be considered as an integer or zero.

Lemma 4.11. *1. For a sequence J with $|J| = q$, we have*

$$\nu^q_{\mathcal{X}}(\phi)_J = \begin{cases} 1, & J = I = \{i_1, \ldots, i_r\} \\ 0, & otherwise \end{cases}$$

2. For a sequence J with $|J| = q+1$,

$$\nu^{q+1}_{\mathcal{X}}(\phi)_J = \begin{cases} i_t, & J = \{i_1, \ldots, i_t+1, \ldots, i_r\},\ t = 1 \ldots r \\ 0, & otherwise \end{cases}$$

Proof. According to Lemma 3.7, integers $\nu^q_{\mathcal{X}}(\phi)_J$ and $\nu^{q+1}_{\mathcal{X}}(\phi)_J$ do not change under the extension of scalars to the field $F(SB(A))$. Therefore we may assume that A is split.

The diagram (4.8) takes the form

$$\begin{tikzcd} M(\mathbb{P}(V))(q)[2q-r] \arrow[r, "\iota"] \arrow[drr, "\psi"'] & M(\mathbf{GL}_{\mathbb{P}(V)}(\mathcal{J}^*)) \arrow[r, equal] & M(\mathbb{P}(V) \times GL_n(F)) \arrow[d] \\ & & M(GL_n(F)) \end{tikzcd}$$

and for each $q \leq k \leq q+n-1$ the morphism ψ gives rise to a morphism of slices

$$\nu^k_{\mathcal{X}}(\psi) : \mathbb{Z}(k)[2k-r] \to \bigoplus_{|J|=k} \mathbb{Z}(k)[2k-l(J)].$$

We have $\nu^k_{\mathcal{X}}(M) = \nu^k_{\mathcal{X}}(GL_n(F))$ by Theorem 4.7(2) and also $\nu^k_{\mathcal{X}}(\phi)$ is equal to $\nu^k_{\mathcal{X}}(\psi)$. The component $\nu^k_{\mathcal{X}}(\psi)_J$ can be non-zero only for J with $l(J) = l(I) = r$, and for such J it can be computed as follows. Consider the induced morphism on motivic cohomology:

$$\psi^* : H^{*,*}(GL_n(F)) \to H^{*-(2q-r),*-q}(\mathbb{P}(V)).$$

Let $h = c_1(\mathcal{O}(1)) \in CH^1(\mathbb{P}(V))$; then

$$\psi^*(c_J(\alpha_0)) = \sum_{k \geq q} \nu^k_{\mathcal{X}}(\psi)_J \cdot h^{k-q} \in CH^*(\mathbb{P}(V)). \tag{4.9}$$

By Proposition 4.3 motivic cohomology $H^{*,*}(\mathbf{GL}_{\mathbb{P}(V)}(\mathcal{J}^*))$ considered as a module over $H^{*,*}(\mathbb{P}(V))$ is free and both

$$\{c_J(\alpha_{\mathcal{J}^*})\}_J$$

and

$$\{c_J(p_2^*\alpha_0)\}_J$$

are bases for this module. Note that by Proposition 2.8 we have $[p_2^*(\alpha_0)] = [p_1^*(\mathcal{O}(-1))] \cdot [\alpha_{\mathcal{J}^*}]$ and multiplicativity formula of the higher Chern classes (4.2) will give the transformation matrix between the two bases above. In particular from

$$c_{j_t}(p_2^*(\alpha_0)) \equiv c_{j_t}(\alpha_{\mathcal{J}^*}) + (j_t - 1)h\, c_{j_t-1}(\alpha_{\mathcal{J}^*}) \ (mod\ h^2)$$

we see that for $J = \{j_1, \ldots, j_r\}$ we have

$$\begin{aligned} c_J(p_2^*(\alpha_0)) = \prod_{t=1}^{r} c_{j_t}(p_2^*(\alpha_0)) \equiv \\ &\equiv \prod_{t=1}^{r} \bigl(c_{j_t}(\alpha_{\mathcal{J}^*}) + (j_t-1)h\, c_{j_t-1}(\alpha_{\mathcal{J}^*})\bigr) \ (mod\ h^2) \equiv \\ &\equiv c_J(\alpha_{\mathcal{J}^*}) + \sum_{t=1}^{r} (j_t-1)h\, c_{j_1,\ldots,j_t-1,\ldots,j_r}(\alpha_{\mathcal{J}^*}) \ (mod\ h^2). \end{aligned}$$

Therefore

$$\psi^*(c_J(\alpha_0)) = \iota^*(c_J(p_2^*(\alpha_0))) \equiv \left\{ \begin{array}{ll} 1, & J = I \\ i_t h, & J = \{i_1, \ldots, i_t + 1, \ldots, i_r\},\ t = 1 \ldots r \\ 0, & otherwise \end{array} \right\} (mod\ h^2),$$

which together with (4.9) gives the desired result. □

We consider the commutative diagram of the connecting morphisms (3.8) in the slice filtrations:

$$\begin{tikzcd}
\mathbb{Z}_{\mathcal{X}}(q)[2q-r] \arrow[r, "\partial'_q"'] \arrow[d, "\nu^q_{\mathcal{X}}(\phi)"'] & \mathbb{Z}_{\mathcal{X}}(q+1)[2q+3-r] \arrow[d, "\nu^{q+1}_{\mathcal{X}}(\phi)"'] \\
\bigoplus_{|J|=q} \mathbb{Z}_{\mathcal{X}}(q)[2q-l(J)] \arrow[r, "\partial_q"'] & \bigoplus_{|J|=q+1} \mathbb{Z}_{\mathcal{X}}(q+1)[2q+3-l(J)]
\end{tikzcd}$$

From the first claim of Lemma 4.11 it follows that the left vertical map is the canonical embedding corresponding to $J = I$. Now we find that

$$\partial_{I,J} = \nu^{q+1}_{\mathcal{X}}(\phi)_J \circ \partial'_q, \tag{4.10}$$

where $\nu^{q+1}_{\mathcal{X}}(\phi)_J$ is determined in the second claim of Lemma 4.11. The class ∂'_q sits in $Hom(\mathbb{Z}_{\mathcal{X}}(q)[2q-r], \mathbb{Z}_{\mathcal{X}}(q+1)[2q+3-r]) = H^{3,1}(\mathbb{Z}_{\mathcal{X}})$. If A splits, then the latter group is zero by Lemma 4.8 and therefore (4.10) implies that $\partial_{I,J} = 0$.

If A does not split, then by Lemma 4.8 the class ∂'_q must be of the form

$$\partial'_q = c_q \cdot \delta,\ c_q \in \mathbb{Z}/n.$$

The arrow ∂'_q which is q-th connecting morphism (3.8) in the slice filtration of $M(SB(A))(q)[2q-r]$ is equal to $\partial'(q)[2q-r]$ where ∂' is the 0-th connecting morphism for the slice filtration of $M(SB(A))$. Both morphisms ∂'_q and ∂' define the same element $c \cdot \delta \in H^{3,1}(\mathbb{Z}_{\mathcal{X}})$ which shows that in fact $c_q = c$ is independent of q.

Lemma 4.12 ([S])**.** *$c \in \mathbb{Z}/n$ is non-zero if A is not split.*

Proof. We exploit the slice spectral sequence (3.7) for $M(SB(A))$ and weight $j = 1$ which has the E_2 term of the following form:

$$\begin{tikzcd}
1 & 0 & \mathbb{Z} \arrow[rrd, "d_2"] & 0 & 0 \\
0 & 0 & F^* & 0 & \mathbb{Z}/n \\
q/p & 0 & 1 & 2 & 3
\end{tikzcd}$$

The connecting morphism $\partial' = c \cdot \delta$ is responsible for the second differential d_2. If $c = 0$, then the spectral sequence degenerates implying that the extension of scalars map

$$CH^1(SB(A)) \to CH^1(\mathbb{P}^{n-1}) = \mathbb{Z}$$

to a splitting field of A is an isomorphism. The Picard-Brauer exact sequence shows that this can not happen unless A is split (see [S84], Theorem 10.12 for a more general result). □

Putting together (4.10), the second claim of Lemma 4.11 and Lemma 4.12 we obtain the desired description of the differential.

□

We would like to use the slice spectral sequence to compute motivic cohomology of $GL_1(A)$ for small weights. In order to do so we need to know the corresponding motivic cohomology groups of $\mathcal{X}$. These have been computed by Merkurjev and Suslin [MS] for the Čech simplicial scheme for any Rost variety X_θ. We apply the results of [MS] when $\theta = (\chi, a) = \chi \cup (a) \in {}_nBr(F) = H^2_{et}(F, \mu_n)$, $\chi \in H^1_{et}(F, \mathbb{Z}/n) = Hom(Gal(F^{sep}/F), \mathbb{Z}/n)$, $a \in H^1_{et}(F, \mu_n) = F^*/(F^*)^n$. In what follows we assume that A is non-trivial a cyclic algebra (χ, a)

We follow [MS] in using the notation

$$H^{*,*}(\mathcal{X})^{\geq 0} := \bigoplus_{p-q-1\geq 0} H^{p,q}(\mathcal{X}),$$
$$H^{*,*}(\mathcal{X})^{\leq 0} := \bigoplus_{p-q-1\leq 0} H^{p,q}(\mathcal{X})$$

and

$$K^\theta_j(F) = coker(\bigoplus_E K^M_j(E) \to K^M_j(F))$$

where K^M_j is the Milnor K-theory functor and the direct sum is taken over all finite extensions E/F that split A. For example we have

$$K^\theta_0(F) = \mathbb{Z}/n\mathbb{Z}$$
$$K^\theta_1(F) = F^*/Nrd(A^*).$$

(for the second statement see [GS], Proposition 2.6.4 and Exercise 2.8).

There is a natural $K^\theta_*(F)$-module structure on $H^{*,*}(\mathcal{X})^{\geq 0}$ ([MS], Proposition 1.2). The Proposition below is a reformulation of [MS], Theorem 1.15 in the case $(X_\theta, n, l) = (SB(A), 2, n)$.

Proposition 4.13. *We have a canonical isomorphism*

$$H^{*,*}(\mathcal{X})^{\leq 0} = H^{*,*}(F)^{\leq 0}$$

and a direct sum decomposition

$$H^{*,*}(\mathcal{X})^{\geq 0} = \bigoplus_{i,k\geq 0} K^\theta_i(F)\cdot\gamma^k\,\delta \oplus \bigoplus_{i,k\geq 0} K^\theta_i(F)\cdot\gamma^{k+1}$$

where $\delta \in H^{3,1}(\mathcal{X})$, $\gamma \in H^{2n+2,n}(\mathcal{X})$ are defined in [MS], 1.6. The bidegree of $K^\theta_i(F)\cdot\gamma^k\,\delta$ is $(i+2k(n+1)+3, i+kn+1)$ and the bidegree of $K^\theta_i(F)\cdot\gamma^{k+1}$ is $(i+2(k+1)(n+1), i+(k+1)n)$.

Corollary 4.14. *In weights $0,1,2$ we have*

$$H^{p,0}(\mathcal{X}) = \begin{cases} \mathbb{Z}, & p=0 \\ 0, & \text{otherwise} \end{cases}$$

$$H^{p,1}(\mathcal{X}) = \begin{cases} F^*, & p=1 \\ \mathbb{Z}/n \cdot \delta, & p=3 \\ 0, & otherwise \end{cases}$$

$$H^{p,2}(\mathcal{X}) = \begin{cases} H^{p,2}(F), & p \leq 2 \\ F^*/Nrd(A^*) \cdot \delta, & p=4 \\ 0, & otherwise \end{cases}$$

Proof. First note that $H^{p,q}(\mathcal{X}) = H^{p,q}(F)$ for $p \leq q+1$, this gives $H^{0,0}$, $H^{1,0}$, $H^{0,1}$, $H^{1,1}$, $H^{2,1}$, $H^{0,2}$, $H^{1,2}$, $H^{2,2}$, $H^{3,2}$.

Let $p > q+1$.

Weight 0: $H(\mathcal{X})^{\geq 0}$ does not contribute since $i+kn+1, i+(k+1)n > 0$ for all $i,k \geq 0$. (Alternatively we could argue using Lemma 3.7.)

Weight 1: $K_i^\theta(F) \cdot \gamma^{k+1}$ does not contribute since $i+(k+1)n \geq n > 1$. $K_i^\theta(F) \cdot \gamma^k\,\delta$ has weight 1 when $i=k=0$, thus giving

$$H^{3,1}(\mathcal{X}) = K_0^\theta(F) \cdot \delta.$$

Weight 2: $i+kn+1=2$ implies $(i,k)=(1,0)$ thus giving

$$H^{4,2}(\mathcal{X}) = K_1^\theta(F) \cdot \delta = F^*/Nrd(A^*) \cdot \delta$$

and $i+(k+1)n=2$ is not possible since $n \geq 3$. □

Remark 4.15. Recall that in this section we assume that n is an odd prime. If $n=2$, then in addition to cohomology groups in weight two listed in the Corollary there is also

$$H^{6,2}(\mathcal{X}) = K_0^\theta(F) \cdot \gamma = \mathbb{Z}/2 \cdot \gamma$$

which appears when $(i,k)=(0,0)$ so that $i+(k+1)n=2$.

Corollary 4.16. *Assume that A is a cyclic algebra of prime odd degree n given by the symbol θ. Motivic cohomology of $GL_1(A)$ of weights 1, 2 and 3 are given as:*

$$\widetilde{H}^{p,1}(GL_1(A)) = \begin{cases} \mathbb{Z}, & p=1 \\ 0, & otherwise \end{cases}$$

$$\widetilde{H}^{p,2}(GL_1(A)) = \begin{cases} F^*, & p=2 \\ n\mathbb{Z}, & p=3 \\ 0, & otherwise \end{cases}$$

$$\widetilde{H}^{p,3}(GL_1(A)) = \begin{cases} H^{0,2}(F), & p=1 \\ H^{1,2}(F), & p=2 \\ H^{2,2}(F), & p=3 \\ \mathbb{Z} \oplus Nrd(A^*), & p=4 \\ n\mathbb{Z}, & p=5 \\ 0, & otherwise \end{cases}$$

Here by $n\mathbb{Z}$ we mean that the extension of scalars to the splitting field for the corresponding motivic cohomology group is injective and the image is $n\mathbb{Z} \subset \mathbb{Z}$.

Proof. In weight j the spectral sequence has nonzero terms

$$E_2^{p,q} = H^{p+q}(\nu^q_{\mathcal{X}}(M), \mathbb{Z}(j))$$

only for $0 < q \leq j$. Let us consider the weights $j = 1, 2, 3$. In these weights the spectral sequence converges to $\widetilde{H}^{*,j}(GL_1(A))$ by theorem 4.7(1). The first three slices of the slice filtration are given by:

$$\begin{aligned}
\nu^1_{\mathcal{X}}(M) &= \mathbb{Z}_{\mathcal{X}}(1)[1] \\
\nu^2_{\mathcal{X}}(M) &= \mathbb{Z}_{\mathcal{X}}(2)[3] \\
\nu^3_{\mathcal{X}}(M) &= \mathbb{Z}_{\mathcal{X}}(3)[4] \oplus \mathbb{Z}_{\mathcal{X}}(3)[5].
\end{aligned}$$

In weight $j = 1$ the slice spectral sequence consists of one row which contains a unique non-zero term $E_2^{0,1} = H^{0,0}(\mathcal{X}) = \mathbb{Z}$, hence we get the isomorphism

$$\widetilde{H}^{1,1}(GL_1(A)) = \mathbb{Z}$$

and the rest of the reduced cohomology groups of $GL_1(A)$ of weight 1 vanish.

In weight $j = 2$ we have two nonzero rows:

$$\begin{aligned}
E_2^{p,1} &= H^{p+1,2}(\mathbb{Z}_{\mathcal{X}}(1)[1]) = H^{p,1}(\mathcal{X}) \\
E_2^{p,2} &= H^{p+2,2}(\mathbb{Z}_{\mathcal{X}}(2)[3]) = H^{p-1,0}(\mathcal{X})
\end{aligned}$$

$$\begin{tikzcd}
2 & 0 & \mathbb{Z} \arrow[rrd, "d_2"] & 0 & 0 \\
1 & 0 & F^* & 0 & \mathbb{Z}/n \cdot \delta \\
0 & 0 & 0 & 0 & 0 \\
{}^{q}/_{p} & 0 & 1 & 2 & 3
\end{tikzcd}$$

and the differential d_2 is multiplication by c which is prime to n by Theorem 4.9. Thus we have

$$\widetilde{H}^{2,2}(GL_1(A)) = F^*$$

$$\widetilde{H}^{3,2}(GL_1(A)) = n\mathbb{Z}$$

and the rest of the reduced cohomology groups of $GL_1(A)$ of weight 2 vanish.

In weight $j = 3$ we have three nonzero rows:

$$\begin{aligned}
E_2^{p,1} &= H^{p+1,3}(\mathbb{Z}_{\mathcal{X}}(1)[1]) = H^{p,2}(\mathcal{X}) \\
E_2^{p,2} &= H^{p+2,3}(\mathbb{Z}_{\mathcal{X}}(2)[3]) = H^{p-1,1}(\mathcal{X}) \\
E_2^{p,3} &= H^{p+3,3}(\mathbb{Z}_{\mathcal{X}}(3)[4] \oplus \mathbb{Z}_{\mathcal{X}}(3)[5]) = H^{p-1,0}(\mathcal{X}) \oplus H^{p-2,0}(\mathcal{X}).
\end{aligned}$$

$$
\begin{tikzcd}
3 & 0 & \mathbb{Z} & \mathbb{Z} \arrow[drr, "d_2"] & 0 & 0 \\
2 & 0 & 0 & F^* \arrow[drr, "d_2"] & 0 & H^{3,1}(\mathcal{X}) \\
1 & H^{0,2}(F) & H^{1,2}(F) & H^{2,2}(F) & 0 & H^{4,2}(\mathcal{X}) \\
0 & 0 & 0 & 0 & 0 & 0 \\
{}^{q}/_{p} & 0 & 1 & 2 & 3 & 4
\end{tikzcd}
$$

By Theorem 4.9 the differential

$$d_2 : \mathbb{Z} = H^{0,0}(\mathcal{X}) \to H^{3,1}(\mathcal{X}) = \mathbb{Z}/n \cdot \delta$$

maps $k \in \mathbb{Z}$ to $\overline{2kc} \cdot \delta$, and since $2c$ is prime to n, the differential is surjective and its kernel is $n\mathbb{Z} \subset \mathbb{Z}$.

Similarly the differential

$$d_2 : F^* = H^{1,1}(\mathcal{X}) \to H^{4,2}(\mathcal{X}) = F^*/Nrd(A^*) \cdot \delta$$

maps $u \in F^*$ to $u^c \cdot \delta$. Since $(F^*)^n \subset Nrd(A^*)$, and c is prime to n, d_2 is surjective with kernel $Nrd(A^*)$. There are no higher differentials by degree reasons and we get the result.

□

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